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Research Article

Comparison of Numerical Techniques for the solutions of Nonlinear Singular Differential Equations

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Abstract

In this research work, the approximate solutions for nonlinear singular initial value problems are to be calculated by using modified Laplace decomposition (MLDM). The modified Adomian decomposition method and Homotopy perturbation methods are to be used to calculate the approximate solutions for the same problem. Moreover, the convergence analysis and the error found for the approximate solution are to be discussed. To prove the robustness and effectiveness of the proposed method, several examples are to be considered and the results calculated in this way will be compared with those obtained from other two above mentioned method. The time consuming behaviour of these method will be noted in order to check the simplicity, accuracy and efficiency of method.

Keywords: *Initial Value Problems, Boundary Value Problems, Modified Laplace Decomposition Method, Homotopy Perturbation Method. He,s polynomials.*

Introduction

Initial value problems of the Lane-Emden type, which can be stated in the subsequent form, can be used as models for an extensive range of problems in the fields of mathematical physics and astronomy. It can be expressed in writing:

$$(f')' + \frac{h}{x} f' + f(x,y)=g(x) \quad 0 \le x \le 1 \tag{1.1}$$

with constraints $y(0)=A$, $y'(0)=B$

where h , x > 0 and f(x, y) is a continuous real-value function and g(x) is an analytical function. This equation is helpful for studying various designs. The numerical solution of LE problem is not easy because of the singularity behavior at origin. The approximate solution of the LE equation is given by Adomian decomposition, homotopy perturbation, variational iteration, differential transformation, wavelets-collocation methods and so forth. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike perturbation theories is not based on the existence of some small parameters.

Wazwaz has given a general way to construct exact and series solutions to Lane-Emden equations by employing the Adomian decomposition method. The paper introduces an approximate approach for solving the Lane-Emden equation accurately. The following work proposes the Homotopy Perturbation Method (HPM) with Laplace Transform (LT) to solve the general kind of Lane-Emden differential equations: The "Preliminaries" section contains information on the homotopy perturbation approach. In the "Results and discussion" section, a few instances of a several types are provided. The conclusion is eventually discussed in the section titled "Findings."

Vibrational Iteration Method

Many scholars have used the vibrational iteration method, first proposed by J. H. He,to solve various linear and nonlinear models. The method's basic idea is to build a correction functional using a general Lagrange multiplier, and to select the multiplier so that its correction solution is better than the initial approximation. Now, in order to demonstrate the method's fundamental idea, we will look at the general nonlinear differential equation presented below in the following form:

$$Pu(x) + Qu(x) = g(x) \tag{1.2}$$

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Where P is a linear operator, Q is a nonlinear operator and g(x) is a known analytical function. We can construct a correction functional according to the variational method as:

$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Pu_n(\xi) + Qu_n(\xi) - g(\xi))d(\xi)$,, $n \geq 0$ 1.3

When, using variational theory, is a generic Lagrange multiplier, the subscript n designates the nth approximation, and un is regarded as a confined variation, with u n = 0. The derived Lagrange multiplier will be applied along with a carefully selected initial approximation, u0, to produce successive approximations, un+1(x). The method's applicability to singular initial value issues of the Lane-Emden type is demonstrated in the examples that follow.

Adomian Decomposition Method

For both linear and nonlinear differential equations, integral equations, and integro-differential equations, the Adomian decomposition method appears to be effective. Adomian first described the approach in his books and other relevant research publications in the early 1990s. Essentially, the approach is a power series approach akin to the perturbation technique. It consists of summarising an infinite number of components that are defined by decomposition series, such as equation 1, from the unknown function u(x) of any equation. In order to illustrate the technique, we will express u(x) as a

$u(x) = \sum_0^\infty u_n(x)$ (1.4)

Or likewise

$u(x) = u_0(x) + u_1(x) + u_2(x) + \ldots\ldots\ldots\ldots$

Where the component $u_{n(x)}$, $n \geq 0$ are to be determined in a recursive manner.

with $u_0(x)$ as the term external the integral sign.

The integral equation is

$u(x) = f(x) + \lambda \int_0^x k(x,t)_u(t)dt$ (1.5)

Hence

$u_0(x) = f(x)$. (1.6)

putting equation (1.4) into equation (1.5)then

$\sum_{n=0}^\infty u_n(x) = f(x) + \lambda \int_0^x k(x,t) \{\sum_{n=0}^\infty u_n t\} dt$ (1.7)

If we indicate the parameters u0(x), u1(x), u2(x),... un(x)..., we are able to find out the components of the unknown function u(x) in an on-going way.
Then from (2.7)

$u_1(x) = \lambda \int_0^x k(x,t)\ u_0(t)dt$
$u_2(x) = \lambda \int_0^x k(x,t)u_1(t)dt$
$u_3(x) = \lambda \int_0^x k(x,t)u_2(t)dt$

$U_n(x) = \lambda \int_0^x k(x,t)\, u_{n-1}(t)dt$ (1.8)
and so on.

This set of equations (5) can be written in compact recurrence scheme as

$u_0(x) = f(x)$.
$u_{n+1}(x) = \lambda \int_0^x k(x,t)u_n(t)dt$ (1.9)

It is important to note that many components may not be able to have the kernel integrated.

Then, in order to approximate the function u, we truncate the series at a specific point (x). Another issue that requires consideration is the convergence of the infinite series solution. Many past workers in this area have tackled this issue. Therefore, it won't be said again here. We'll use various examples to show the technique.

Analysis of Lane–Emden type Equation

This part, correction functional is constructed and the solution to Eq. (1.4) is expressed in the form of He's variational iteration.

$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t) \left[(Y_n)tt + \frac{2}{t}(y_n)t + f(t,y_n) - g(t)]dt\right]$ (1.10)

To determine the optimal value of λ(s), we continue as follows;

$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(t) \left[(Y_n)tt + \frac{2}{t}(y_n)t + f(t,y_n)g(t)]dt\right]$ (1.11)
$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(t) \left[(Y_n)tt + \frac{2}{t}(y_n)t +]dt\right]$ (1.12)
which gives
$\delta y_{n+1}(x) = [1 - \lambda'(x) + \frac{2}{x}\lambda(x)] \cdot \delta y_n(x) + \delta\lambda(x) \cdot (y_n)t(x) + \int_0^x \delta y_n\ [\lambda''(t) - 2t\lambda'(t) - \frac{\lambda(t)}{t^2}]dt = 0.$ (1.12)

Hence, the stationary conditions can be obtained from Eq. (2.12) as;

$1 - \lambda'(x) + \frac{2}{x}\lambda(x) = 0, \quad \lambda(x) = 0 \quad \lambda''(x) - 2\frac{x\lambda'(x) - \lambda(x)}{x^2} = 0.$ (1.13)

When it is found that the Lagrange multiplier is
$\lambda(t) = \frac{t^2}{x} - t$
Finally, the iteration formula can be found as:

$y_{n+1}(x) = y_n(x) + \int_0^x [\frac{t^2}{x} - t]\,(Y_n)\,tt\,\frac{2}{t}\,(y_n)t + f(t, y_n) - g(t)]dt]$

Modified Decomposition Method (MDM)

$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt$

The function f(x) can be defined in this approach as the sum of two partial functions, f1(x) and f2(x). To put it another way, we can

$f(x) = f(x_1) + f(x_2)$

This introduces the recurrence relation

$u_0(x) = f_1(x)$
$u_1(x) = f_1(x) + \lambda \int_0^x k(x\ t)u0tdt$
$u_{k+1} = \lambda \int_0^x k(x,t)uktdt$

HE'S Homotopy Perturbation Method

This approach is very beneficial for resolving nonlinear equations.

Think about the common non-linear equation $A(u) - \int(r) = 0, r \in \Omega$ with condition

$A(x) - \int(y) = 0, r \in \Omega$
$A(x) - \left(u, \frac{\partial x}{\partial \pi}\right) 0, r \in \mathrm{r}\,\Omega$

Where A is general differential. B a boundary operator, f(y) a known analytic function and T is the boundary Ω. The operator A can be replaced by L and N. wher L a linear and N is nonlinear. Therefore

$A(x) - f(y) = 0$
Becomes
$L(x) + N(x) - f(x) = 0,$
(2.1)
Using Homotopy, we define $v(y,p): \Omega \times [0.1] \rightarrow R\ as$
$H(v,P) = (1-P)[L(v) - L(x_0)] + P\,[A(v) - f(y)] = 0$ (2.2)
$P \in [0.1]$
Or
$H(v, P0 = L(v) - L(x_0) + PL(x_0) + P[N(v) - f(y)] = 0$ (2.3)
$Where\ p \in [0.1] and\ u_0$ is the initial approximation $of\ A(u) - f(r) = 0$ Satisfy the
Given condition Clearly,
$H(v,o) = L(v) - L(x_0) = 0$ (2.4)
$H(v,1) = A(v) - f(y) = 0$ (2.5)

Comparable to how v(r,p) changes from v 0 (r) to u is the change of p from zero to unity (r). L(v) -L(u 0) and A(v) -f(r) are homotopic in this deformation. The answer to (1.12) and (1.13) has the following form if p is a small parameter:

$v = v_0 + pv_1 + p^2 v_2 + \cdots$ (2.6)
$For, p = 1, we\ have$ (2.7)
$u = lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$

It is established that (2.7) converges. Implying this approach because homotopy makes it simple to choose an initial approximation, the perturbation equation can be constructed in a variety of ways. Homotopy also plays a significant role in providing the necessary precision.

Example 1:
Consider non-linear BV problem
$y'' - \frac{1}{x}y' = \frac{x^2}{3}y^5$ (3.1)
$y(0) = 1, y'(1) = \frac{\sqrt{3}}{8}$ (3.2)
(a) Solution using MADM:
We define $L_1 = \frac{x^3 d}{dx}\left(\frac{x^{-1}d}{dx}\right)$
$L_1^{-1}(.) = \int_0^x x \int_1^x x^{-3}\,(.)dx\,dx$
Equation (3.1) becomes
$L_1 y = \frac{x^4}{3}y^5$
Applying L_1^{-1}on the sides
$L_1^{-1}L^1 y = \frac{L_1^{-1}x^4}{3}y^5$
$\int_0^x x \int_1^x x^{-3}\left[x^3 \frac{d}{dx}\left(x^{-1}\frac{dy}{dx}\right)\right] dx\ \ dx = L_1^{-1}\ \frac{x^4}{3}y^5$
$\int_0^x x\,|x^{-1}y'|_1^x dx = \frac{L_1^{-1}x^4}{3}y^5$
$\int_0^x x\,|[x^{-1}y'(x) - y'(1)]|_1^x dx = L_1^{-1}\frac{x^4}{3}y^5$
$|y(x)|_0^x - \left|\frac{x^2}{2}y'(1)\right|_0^x = L_1^{-1}\ \frac{x^4}{3}y^5$
$y(x) - y(0) - \frac{x^2}{2}y'(1) = L_1^{-1}\frac{x^4}{3}y^5$
$y(x) = y(0) + \frac{1}{2}y'(1)x^2 + \frac{L_1^{-1}x^4}{3}y^5$
For linear function we use decomposition series.
$y_n(x) = \sum_{n=0}^{\infty} y_n$
For y^5 using polynomial series where y^5 is nonlinear
$F(y) = \sum_{n=0}^{\infty} y_n$
$\sum_{n=0}^{\infty} y_n = y(0) + \frac{1}{2}y'(1)x^2 + L_1^{-1}\left(\frac{x^4}{3}\sum_{n=0}^{\infty} A_n\right)$
This gives
$y_0(x) = y(o) + \frac{1}{2}y'(1)x^2$
Using (3.2)
$y_0(x) = 1 - \frac{\sqrt{3}}{16}x^2 = 1 - .108253x^2$
$y_{n+1} = L_1^{-1}A_n \qquad n \geq 0$ (3.3)
Now for n=0
$y_1 = L_1^{-1}\left(\frac{x^4}{3}A_0\right)$
$A_0 = F(y_0) = y_0^5$
$A_0 = \left(1 - \frac{\sqrt{3}}{16}x^2\right)^5$
$A_0 = L_1^{-1}\ \frac{x^4}{3}\ \left(1 - \frac{\sqrt{3}}{16}x^2\right)^5$
$y_1 = \frac{1}{3}\int_0^x x \int_1^x x^{-3}[x^4(1 = 0.108253x^2)^5 dxdx$
$y_1 = \frac{(1-0.108253x^2)^7}{5.9062231} + .0114411x^2$
$y_1 = -0.0637827x^2 + .416666x^4 - .0075176x^6 + .00081380x^8 - .00005286x^{10} + 1.9073486 \times 10^{-6}x^{12} - 2.9496649 \times 10^{-8}x^{14}$

$y_2 = L_1^{-1}\left(\frac{x^4}{3}A_1\right)$

$A_1 = y_1F'(y_0) = y_1.5y_0^4 = 5y_0^5y_1$

$y_2 = .0062577x^2 - .0044294x^6 + .0024057x^8 - .0006259x^{10} + .0001017x^{12} - .0000114x^{14} + \cdots,$

$y_2 = -.0010668x^2 + .000436x^6 + .0001884x^8 - .0003595x^{10} + .0001864x^{12} - .0000553x^{14} + \cdots,$

So the Approximate solution is:

$y = y_0 + y_1 + y_2 + y_3$

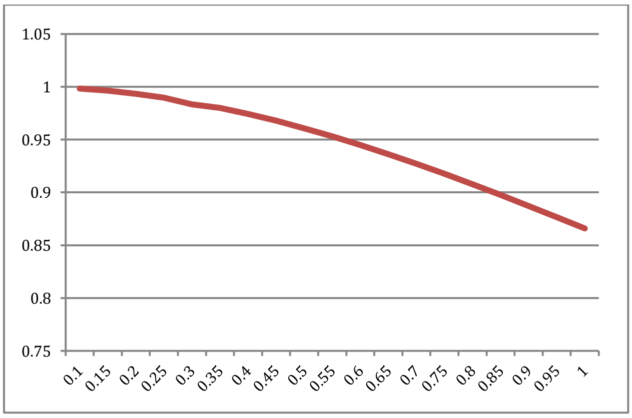
$y = 1 - .1668450x^2 + .0416667x^4 - .0115124x^6 + .0034079x^8 - .0010384x^{10} + .0002900x^{12} - .0000668x^{14} + \cdots$

Summary results by MLDM

Table 3.1 Summary of the Results from Example 1

X	Exact	Approximate	Error
.1	.998337488459588	0.998335705191570	1.7882688080E-05
.15	.996270962773433	0.996270962773438	4.0917585350E-05
.2	.100 400268898783	0.993392138545470	6.1302544139E-05
.25	.989743318610788	0.989732188429514	1.1340181275E-06
.3	.1000329277764294	0.983313275341990	1.5992822399E-06
.35	.980188050780010	0.980166325291873	2.172588135E-06
.4	.974354703692116	0.974326441932890	2.8871579776E-04
.45	.967867833691654	0.967832282253857	3.6004737800E-03
.5	.960768922830528	0.960725402270505	4.5320569915E-05
.55	.953101634249033	0.953049592551692	5.2142697351E-05
.6	.944911182523070	0.944850221531165	6.0960991904E-05
.65	.936243679766950	0.936173601673489	7.0077093473E-06
.7	.927145540823121	0.927066389666090	7.8991158021E-03
.75	.917662935482248	0.917575026686240	8.8008797706E-05
.8	.907841299003203	0.907745217880645	9.5991122560E-05
.85	.897724905923811	0.897621440407650	1.0436551616E-03
.9	.887356509416115	0.887246454882580	1.1905453453E-05
.95	.876777046043595	0.876660772972148	1.2628307145E-04
1	.8660254037844340	0.865919000000000	1.2340378444E-04

Precise and approximate solutions are compared



X-axis

Figure 3.1: Graphical Comparison of Exact and Approximate Solution of Example 1

Example 2:

Let's say the BV Problem is nonlinear.

$$y^{-1} + \frac{3}{x}y'' - y^3 = g(x) \tag{3.4}$$

$$y(0) = 0, y' = 0, y(1) = e, \tag{3.5}$$

Where $g(x) = 24e^x + 35xe^x + 12x^2e^x + x^3e^x - x^9e^{3x}$.

Using the $T-$series of $g(x)$with order 10.

$g(x) = g(T) = 24 + 60x + 60x^2 + 35x^3 + 14x^4 + \frac{21}{5}x^5 + x^6 + \frac{11}{56}x^7 + \frac{11}{336}x^8 - \frac{30097}{30240}x^9 - \frac{64787}{21600}x^{10}$

(a) Solution using MADM:

We define

$L_1 = x\frac{d}{dx}\left(x\frac{d}{dx}\right),$

$L_2 = x^{-2}\frac{d}{dx'}$

$L_1^{-1} = \int_0^x x^{-1}\int_0^x x^{-1}(.)dx.$

$$L_1y = gT(x) + L_2y + y^3 \tag{3.6}$$

$$r) = L_1^{-1}L_2^{-1}gT\ (x) + L_1^{-1} + L1^{-1}L_2^{-1}y^3 \tag{3.7}$$

Using polynomial series for y_3 and decomposition series for $y(x)$, we obtain

$y_n(x) = \sum_{n=0}^{\infty} y_n$

$F(y) = \sum_{n=0}^{\infty} A_n$

$y_n(x) = \sum_{n=0}^{\infty} y_n = L_1^{-1}L_2^{-1}gT(x) + L_1^{-1}(\sum_{n=0}^{\infty} y_n) + l_1^{-1}L_2^{-1}(\sum_{n=0}^{\infty} A_n)$

Its provides recursive relation

$g(x) = L_1^{-1}L_2^{-1}gT(x)$

$y_{k+1}(x) = L_1^{-1}y_k + l_1^{-1}L_2^{-1}A_k, k \geq 0$

The Adomian polynomials for $y^3$are computed as

$sA_0 = y_0^3,$

$A_1 = 3y_0^2y_1$

$A_2 = 3y_2y_0^2 + 3y_0y_1^2$

$A_3 = 3y_3y_0^2 + 6y_0y_1y_2 + y_1^3$

Substituting L_1^{-1}, L_2^{-1} into (3.7) The ADM leads to the following scheme:

$\int_1^x x^{-1}\int_0^x x^{-1}\int_0^x x^2 \quad 24 + 60x + 60x^2 + 35x^3 + 14x^4 + \frac{21}{5}x^5 + x^6 + \frac{11}{56}x^7 + \frac{11}{336}x^8 - \frac{30097}{30240}x^9 - \frac{64787}{21600}x^{10})\ dxdxdx$

$0 = 0.88888889x^3 + 0.9375000x^4 + 0.4800000x^5 + 0.1620370x^6 + 0.0408163x^7 + 0.0082031x^8 + 0.0013717x^9 + 0.0001964x^{10} + 0.0000246x^{11} - 0.0005759x^{12} - 0.0013652x^{13}$

Using Mat lab next iterations are

$0 = -0.0013652\hat{}*X\hat{}13 - 0.0005797\hat{}*X\hat{}12 + 0.000024597\hat{}*X\hat{}11 + 0.00019643\hat{}*X\hat{}10 + .0013717\hat{}*X\hat{}9 + .00820231\hat{}*X\hat{}8 + .040816\hat{}*X\hat{}7 + .16204\hat{}*X\hat{}6 + .48\hat{}*X\hat{}5 + .9375\hat{}*X\hat{}4 + .88889\hat{}*X\hat{}3$

$Y_1 = 0.00000000000003445^*X^{42} - 0.000000000000000017251^{*39} - 0.0000000000000015707^*X^{38} - 0.000000000000000010364^*X^{37} - 0.00000000000057408^*X^{36} - 0.000000000000000025719^*X^{35} - 0.00000000000087988^*X^{34} - 0.00000000000020773^*X^{33} - 0.000000000000000026549^*X^{32} + 0.000000000000028639 + 0.0000000000000097034^*x^{30} +$

Sauessive iterations give us the following results

$y_2 = 0.0109739x^3 + 0.0036621x^4 + 0.0007680x^5 + 0.0001250x^6 + 0.0000169x^7 + 1.6799819 \times 10^{-9}x^{11} + \cdots,$

$y_{3=}0.0012193x^3 + 0.0002289x^4 + 0.0000307x^5 + 0.00000347x^6 + 3.4693305 \times 10^{-7}x^7 + 3.1292439 \times$

$10^{-8}x\&8 + 2.5811748 \times 10^{-9}x^9 + 1.9642857 \times 10^{-10}x^{10} + 1.3884184 \times 10^{-11}x^{11} + \cdots,$

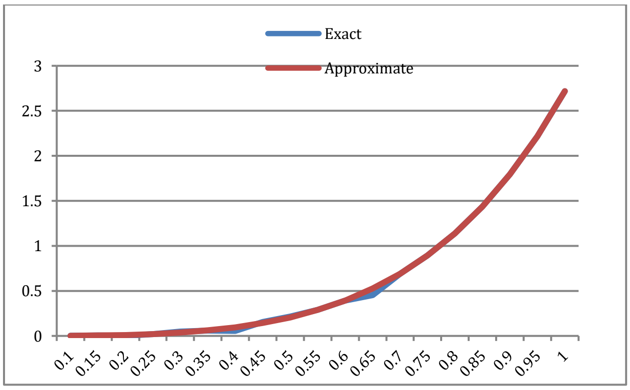
Approximated solution is: $y = y_0 + y_1 + y_2 + y_3$

$y = 9998476x^3 + .999947x^4 + .4999987x^5 + 1.666666x^6 + .0416667x^7 + .0083333x^8 + .0013888x^9 + 0.0001984x^{10} + 0.0000248x^{11} + \cdots$

Table 3.2: Summary of the results from example 2

X	Exact	Approximate	Error
.1	0.001105170918057	0.001105016875013	1.5394306500E-06
.15	0.004022190569288	0.003920668374150	5.2219506000E-06
.2	0.001088222066398	0.009769977965300	1.2440999930E-07
.25	0.019962777135736	0.020060454835328	2/4423004360E-07
.3	0.046448887804772	0.036441945868720	4.2419358440E-07
.35	0.059842251154939	0.060835750446840	6.8806100100E-07
.4	0.055576780849004	0.095466621743900	1.0158905251E-06
.45	0.153012448002792	0.142897908312666	1.4539600116E-05
.5	0.216080158837466	0.206070110156260	2.0048681366E-06
.55	0.288369970847688	0.288343145530644	2.7025417044E-05
.6	0.393577660884350	0.393542648421666	3.5012462795E-06
.65	0.452055488167541	0.526010642095518	4.4758072426-04
.7	0.690717178662373	0.690660961451320	5.6217211056E-07
0.75	0.893109382008472	0.893039826223288	6.9555785289E-04
0.8	1.139476955388140	1.139391998513388	8.4956873760E-07
0.85	1.436835622939040	1.436732991814180	1.0263112585E-07
0.9	1.793050668033410	1.792927834728430	1.2283340507E-06
0.95	2.216922819155922	2.216776931163770	1.4588799226E-07
1	2.718281828459045	2.718109700000000	1.7222875904E-06

Comparison of exact and approximate solutions



X-axis

Conclusions

In this thesis, the Modified Laplace decomposition method (MLDM), modified adomian decomposition method (MADM), and homotopy perturbation method (HPM), have been used to solve the nonlinear singular initial value problems. In this case of nonlinear initial value problems (IVPs), and the boundary value problems (BVPs), the homotopy perturbation is used. While complicated and challenging calculations are overcome by He׳s homotopy perturbation method (HPM). The outcomes obtained due to approximations for higher order nonlinear boundary and initial value problems are compared. It is found that modified laplace decomposition method is a less time consuming tool as compared to modified adomian decomposition and homotopy perturbation method. The comparison of the results obtained using these three methods shows nearly identical findings. The proposed method is also more efficient to overcome the singular behaviour of problems and it illustrates the approximations of high precision with a large effective region of convergence.

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